CHAPTER 3

DETERMINANTS

3.1 INTRODUCTION

Consider two homogeneous linear equations

$$a_1x + b_1y = 0,$$
$$a_2x + b_2y = 0;$$

Multiplying the first equation by b_2, the second by b_1, subtracting and dividing by x, we obtained

$$a_1b_2 - a_2b_1 = 0$$

This result is sometimes written as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

and the expression on the left is called the determinant.

A determinant also is an arrangement of numbers in rows and columns but it always has a square form and can be reduced to a single value. Therefore, a determinant is distinct from matrix in the sense that the determinant is always in square shape and it has a numerical value. The arrangement of the numbers of a determinant is enclosed within two vertical parallel lines.

Order of a Determinant

The determinant of a square matrix of order n is known as determinant of order n.

3.2 DETERMINANT OF ORDER TWO

Let $a_{11}, a_{12}, a_{21}, a_{22}$ be any four number (real or complex). Then

$$|A| = \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix}$$

represent the number $a_{11}\,a_{22} - a_{21}\,a_{12}$ and is called *a* determinant of order two.

For example

$$|A| = \begin{vmatrix} 5 & 2 \\ 3 & -7 \end{vmatrix} = (5)(-7) - (3)(2)$$
$$= -35 - 6 = -41$$

3.3 DETERMINANT OF ORDER THREE

Let $$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is called a determinant of order 3 and its value can be obtained as follows:

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

For example, $$|A| = \begin{vmatrix} 2 & 3 & 5 \\ -1 & 2 & 3 \\ 4 & -2 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} + 4\begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix}$$

$$= 2(2+6) - 3(-1-12) + 4(2-8)$$

$$= 16 + 39 - 24 = 31$$

Remarks

- The value of a determinant is not changed if it is expanded along any row or column.
- When no reference of the corresponding matrix is needed, we may denote a determinant by D.
- The determinant of a square zero matrix is zero.

SOLVED EXAMPLES

Example 1: *Find the value of $|A|$ if A is given by* $\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$

Solution: $$|A| = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$$

$$= \cos^2\alpha - (-\sin^2\alpha) = \cos^2\alpha + \sin^2\alpha = 1.$$

Example 2: *Find the value of* $\begin{vmatrix} 1 & \omega \\ \omega & -\omega \end{vmatrix}$.

Solution: $|A| = \begin{vmatrix} 1 & \omega \\ \omega & -\omega \end{vmatrix} = -\omega - \omega^2 = -(\omega + \omega^2) = (-1) = 1.$

Example 3: *Solve for x :*

$$\begin{vmatrix} x & 3 \\ 5 & 2x \end{vmatrix} = \begin{vmatrix} 5 & -4 \\ 5 & 3 \end{vmatrix}$$

Solution: We have $\begin{vmatrix} x & 3 \\ 5 & 2x \end{vmatrix} = \begin{vmatrix} 5 & -4 \\ 5 & 3 \end{vmatrix}$

$\Rightarrow \quad 2x^2 - 15 = 15 + 20$

$\Rightarrow \quad 2x^2 = 50 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5.$

Example 4: *Find the value of* $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Solution : Let $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

We expand Δ along first row, we get

$$\Delta = a(-1)^2 \begin{vmatrix} b & f \\ f & c \end{vmatrix} + h(-1)^3 \begin{vmatrix} h & f \\ g & c \end{vmatrix} + g(-1)^4 \begin{vmatrix} h & b \\ g & f \end{vmatrix}$$

$$= a(bc - f^2) - h(hc - fg) + g(hf - bg)$$

$$= abc - af^2 - ch^2 + fgh + fgh + fgh - bg^2$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2$$

Example 5: *Find the value of* $\begin{vmatrix} 0 & 1 & \sec\theta \\ \tan\theta & -\sec\theta & \tan\theta \\ 1 & 0 & 1 \end{vmatrix}$

Solution: Let $\Delta = \begin{vmatrix} 0 & 1 & \sec\theta \\ \tan\theta & -\sec\theta & \tan\theta \\ 1 & 0 & 1 \end{vmatrix}$

expand along R_1

$$\Delta = 0 + 1(-1)^3 \begin{vmatrix} \tan\theta & \tan\theta \\ 1 & 1 \end{vmatrix} + \sec\theta(-1)^4 \begin{vmatrix} \tan\theta & -\sec\theta \\ 1 & 0 \end{vmatrix}$$

$$= -(\tan\theta - \tan\theta) + \sec\theta(0 + \sec\theta) = \sec\theta(0 + \sec\theta) = \sec^2\theta.$$

3.4 CO-FACTORS AND MINORS OF AN ELEMENT

If in the expansion of a determinant $|a_{ij}|$, all the containing a_{ij} as a factor, are collected and their sum a is denoted by $a_{ij} A_{ij}$ then the factor A_{ij} is called the co-factor of the element a_{ij}. Hence, in a determinant of order n

$$|a_{ij}| = a_{i1} A_{i1} + a_{i2} A_{i2} + ... + a_{in} A_{in} = \sum_{j-1}^{n} a_{ij} A_{ij}$$

Now, let M_{ij} be the $(n-1) \times (n-1)$ sub-matrix of $|a_{ij}|_{n \times n}$ obtained by deleting the ith row and jth column. Then $|M_{ij}|$ is called the minor of the element a_{ij} the determinant $|a_{ij}|$ of order n. Thus we can express the determinant as a linear combination of the minors of the elements of any row or any column.

Remark

- $(-1)^{t+j}$ is 1 or -1 according as $t+j$ is even or odd

  $\therefore$ A_{ij} and M_{ij} coincides if $i+j$ is even and if $i+j$ is odd then we have $A_{ij} = -M_{ij}$.

SOLVED EXAMPLES

Example 1: *Find the minors and cofactors of elements of the determinant* $\begin{vmatrix} 5 & -2 \\ 3 & 7 \end{vmatrix}$

Solution:

Minor of the element a_{11} is $M_{11} = |7| = 7$

Minor of the element a_{12} is $M_{12} = 3$

Minor of the element a_{21} is $M_{21} = -2$

Minor of the element a_{22} is $M_{22} = 5$

Hence, $A_{11} = (-1)^{1+1} M_{11} = 7$

$A_{12} = (-1)^{1+2} M_{12} = -3$

$A_{21} = (-1)^{2+1} M_{21} = 2$

$A_{22} = (-1)^{2+2} M_{22} = 5$

Example 2: *Find all the minors and cofactors of the elements in following determinants*

$$\begin{vmatrix} 4 & 3 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 5 \end{vmatrix}$$

Solution : Here

$a_{11} = 4 a_{12} = 3 a_{13} = 1$

$a_{21} = 1 a_{22} = 3 a_{23} = 2$

$a_{31} = -2 a_{32} = 1 a_{33} = 5$

$M_{11} = \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = 15 - 2 = 13$ $\qquad M_{12} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1$

$M_{13} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5$ $\qquad M_{21} = \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 15 - 1 = 14$

$M_{22} = \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} = 20 - 2 = 18$ $\qquad M_{23} = \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 4 - 6 = -2$

$M_{31} = \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 6 - 3 = 3$ $\qquad M_{32} = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 8 - 1 = 7$

$M_{33} = \begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix} = 12 - 3 = 9.$

The co-factors are

$A_{11} = (-1)^{1+1} M_{11} = 1 \times 13 = 13$ $\qquad A_{12} = (-1)^{1+2} M_{12} = -1 \times 1 = -1$

$A_{13} = (-1)^{1+3} M_{13} = 1 \times (-5) = -5$ $\qquad A_{21} = (-1)^{2+1} M_{21} = -1 \times 14 = -14$

$A_{22} = (-1)^{2+2} M_{22} = 1 \times 18 = 18$ $\qquad A_{23} = (-1)^{2+3} M_{23} = -1 \times (-2) = 2$

$A_{31} = (-1)^{3+1} M_{31} = 1 \times 3 = 3$ $\qquad A_{32} = (-1)^{3+2} M_{32} = -1 \times 7 = -7$

$A_{33} = (-1)^{3+3} M_{33} = 1 \times 9 = 9.$

Example 3: *Find the minor and co-factors of elements of the following determinant*

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

Solution: We have

$$M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0-20 = -20 \qquad A_{11} = -20$$

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42-4 = -46 \qquad A_{12} = 46$$

$$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30-0 = 30 \qquad A_{13} = 30$$

$$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21-25 = -4 \qquad A_{21} = 4$$

$$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14-5 = -19 \qquad A_{22} = -19$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10+3 = 13 \qquad A_{23} = -13$$

$$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12-0 = -12 \qquad A_{31} = -12$$

$$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8-30 = -22 \qquad A_{32} = 22$$

$$M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0+18 = 18 \qquad A_{33} = -18.$$

Example 4: *Write the co-factors of elements of the second row of the following determinants and hence evaluate them*

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Solution: Let

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} a & bc \\ c & ab \end{vmatrix} = -(a^2b - bc^2)$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & bc \\ 1 & ab \end{vmatrix} = ab - bc$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & a \\ 1 & c \end{vmatrix} = -(c-a) = a-c$$

$$\Delta = -(a^2b - bc^2) + b\,(ab - bc) + ca\,(a-c)$$
$$= bc^2 - a^2b + ab^2 - b^2c + a^2c - ac^2.$$

3.5 PROPERTIES OF DETERMINANTS

Theorem 1: *The value of a determinant does not change when rows and columns are interchanged.*

Proof: Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant of order three.

Expanding $|A|$ along the first row, we get

$$|A| = a_1\,(b_2c_3 - b_3c_2) - b_1\,(a_2c_3 - a_3c_2) + c_1\,(a_2b_3 - a_3b_2)$$
$$= a_1\,(b_2c_3 - b_3c_2) - a_2\,(b_1c_3 - a_3c_1) + a_3\,(b_1c_2 - b_2c_1)$$

(by rearrangement of terms)

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Hence, the theorem is proved.

Theorem 2: *If any two rows [or columns] of a determinant are interchanged, the sign of the determinant is changed.*

Proof: Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant of order three.

Expanding $|A|$ along the first row, we get

$$|A| = a_1\,(b_2c_3 - b_3c_2) - b_1\,(a_2c_3 - a_3c_2) + c_1\,(a_2b_3 - a_3b_2)$$
$$= -\{a_3\,(b_2c_1 - b_1c_2) - b_3\,(a_2c_1 - a_1c_2) + c_3\,(a_2b_1 - a_1b_2)\}$$

(by rearrangement of terms)

$$= -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (-1)\,|A|$$

Theorem 3: *If two rows or two columns of the determinant are identical, then the value of the determinant vanishes, i.e.,*

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0.$$

Proof : We have $|A|$ is a determinant of order 3 whose first and third row are identical. If we interchange the two identical rows, then obviously there will be no change in the value of $|A|$. But by theorem 2, the value of A is multiplied by -1 if we interchange two rows. Therefore, we get

$$|A| = -|A|$$
$$2|A| = 0 \text{ or } |A| = 0$$

Theorem 4: *If all the elements of any row, or any column, of a determinant are multiplied by the same number then the determinant is multiplied by that number.*

Proof : Let $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$ be a determinant of order n

We have $\begin{vmatrix} ma_{11} & a_{12} & \cdots & a_{1n} \\ ma_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ma_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = ma_{i1}A_{i1} + ma_{i2}A_{i2} + \ldots ma_{in}A_{in} = m|A|$

(where $A_{i1}, A_{i2} \ldots A_{in}$ be the cofactor of elements $a_{i1}, a_{i2}, \ldots a_{in}$ of ith row of $|A|$)

Theorem 5: *If in the determinant, the elements of a row are added in and m times the corresponding elements of the another rows (or column), the value of the determinant does not change in particular,*

$$\begin{vmatrix} a_1 + mb_1 + nc_1 & b_1 & c_1 \\ a_2 + mb_2 + nc_2 & b_2 & c_2 \\ a_3 + mb_3 + nc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Proof : We have

$$\begin{vmatrix} a_1 + mb_1 + nc_1 & b_1 & c_1 \\ a_2 + mb_2 + nc_2 & b_2 & c_2 \\ a_3 + mb_3 + nc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} nc_1 & b_1 & c_1 \\ nc_2 & b_2 & c_2 \\ nc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + n\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

(By theorem 4)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m(0) + n(0)$$

(By theorem 3)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

SOLVED EXAMPLES

Example 1: *Evaluate the following determinant*

$$\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix}$$

Solution: We have $|A| = \begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3 \times 5 - 4 \times (-2) = 15 + 8 = 23$

Example 2: *Find the value of the determinant of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution: We have $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

On expanding the determinant along the first row, we get

$$= 1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 1.(6-1) - 2.(4-3) + 3.(2-9) = -18$$

Example 3: *Evaluate the determinant of* $\begin{vmatrix} 4 & 1 & 4 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$.

Solution: We have $|A| = \begin{vmatrix} 4 & 1 & 4 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$

On expanding the determinants along first column, we get

$$= 4\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix}$$

$$= 4(1-0) - 0 + 1(0-4)$$

$$= 4 - 4 = 0.$$

Example 4: *Show that:*

$$\begin{vmatrix} 1 & x & y \\ 0 & \cos x & \sin y \\ 0 & \sin x & \cos y \end{vmatrix} = \cos(x+y)$$

Solution: We have $\begin{vmatrix} 1 & x & y \\ 0 & \cos x & \sin y \\ 0 & \sin x & \cos y \end{vmatrix}$

On expanding the determinant along first column, we get

$$= 1\begin{vmatrix} \cos x & \sin y \\ \sin x & \cos y \end{vmatrix} - 0\begin{vmatrix} x & y \\ \sin x & \sin y \end{vmatrix} + 0\begin{vmatrix} x & y \\ \cos x & \cos y \end{vmatrix}$$

$$= \cos x \cos y - \sin x \sin y = \cos (x + y)$$

Example 5: *Show that* $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$

Solution: We have L.H.S. = $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$

Applying $C_2 - C_1$ and $C_3 - C_1$ in the given determinant, we get

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{vmatrix}$$

On expanding the determinant along the first row, we get

$$= 1\begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 1 & y \end{vmatrix} - 0\begin{vmatrix} 1 & x \\ 1 & 0 \end{vmatrix} = xy = \text{R.H.S.}$$

Example 6: *Without expanding, show that*

$$\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix} = 0$$

Solution: We have

$$\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix} = \begin{vmatrix} 0 & c-a & a-b \\ 0 & a-b & b-c \\ 0 & b-c & c-a \end{vmatrix}$$

(Operating $C_1 \to C_1 + C_2 + C_3$, we get) = 0

Example 7: *Without expanding, show that*

$$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$$

Solution: Consider

$$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$$

(Multiplying R_1 by a, R_2 by b and R_3 by c)

$$= \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ca \\ bc^2a^2 & abc & bc+ab \\ ca^2b^2 & abc & ca+bc \end{vmatrix} \qquad \text{(Take } abc \text{ out from } C_1 \text{ and } C_2\text{)}$$

$$= \frac{abc.abc}{abc}\begin{vmatrix} bc & 1 & ab+ca \\ ca & 1 & bc+ab \\ ca & 1 & ca+bc \end{vmatrix}$$

$$= abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ab \\ cb & 1 & ab+ca+bc \end{vmatrix} \qquad \text{(Operate } C_3 \to C_3 + C_1\text{)}$$

$$= abc\,(ab+bc+ca)\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ cb & 1 & 1 \end{vmatrix}$$

$$= abc\,(ab+bc+ca) \times 0 = 0$$

Example 8: *If a, b, c are in A.P. prove that* $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$

Solution: Given a, b, c are in A.P. therefore $a+c=2b$

$\Rightarrow a+c-2b=0$

Operating $R_1 \to R_1 + R_3 - 2R_2$, we get

$$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = \begin{vmatrix} 0 & 0 & a+c-2b \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$$

Example 9: *Prove that*

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = abc\,(a-b)\,(b-c)\,(c-a)$$

Solution: We have $|A| = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = \text{abc}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

Now again $\qquad |A| = abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

Applying $C_2 - C_1$ and $C_3 - C_1$, we get

$$= abc \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

On expanding along the first row, we get

$$= abc \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$= abc\,[(b-a)(c^2-a^2)-(b^2-a^2)(c-a)]$$
$$= abc\,[(b-a)(c-a)\{(c+a)-(b+a)\}]$$
$$= abc\,(b-a)(c-a)(c+a-b-a)$$
$$= abc\,(a-b)(b-c)(c-a)$$

Example 10: *Prove that*

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2\,(a+b+c)^3$$

Solution: Let $|A| = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Adding C_2 and C_3 in C_1, we get

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2\,(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $(R_2 - R_1)$ and $(R_3 - R_1)$, we get

$$= 2\,(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

On expanding the determinant along the first column, we get

$$= 2\,(a+b+c) \begin{vmatrix} b+c+a & 0 \\ 0 & a+b+c \end{vmatrix}$$

$$= 2\,(a+b+c)(a+b+c)^2$$
$$= 2\,(a+b+c)^3$$

Example 11: *Prove that*

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

[UPTU B. Pharma 2000, 06]

Solution : Operating $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \begin{vmatrix} a & 0 & 1 \\ 0 & b & 1 \\ -c & -c & 1+c \end{vmatrix}$$

$$= a[b.(1+c)-(-c).1] + 1\,[0.(-c)-(-c)\,b]$$

$$= a\,(b + bc + c) + bc$$

$$= abc + bc + ca + ab = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Example 12: *Prove that*

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

Solution: Operating $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

[Take $(a+b+c)$ out from R_1]

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

(Operate $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & a-b-c \end{vmatrix} \text{ (expand by } R_1)$$

$$= (a+b+c)\,1\,(-a-b-c)\,(-a-b-c) = (a+b+c)^3$$

Example 13: *Without expanding the determinant, show that*

$$\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0.$$

[UP TU B. Pharma 2001]

Solution: Let $\Delta = \begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix}$

By changing columns into rows:

$$\Delta = \begin{vmatrix} 0 & -b & c \\ b & 0 & -a \\ -c & a & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix}$$

(taking (–1) Common from each column)

$$= (-1)^3 \Delta = -\Delta.$$

$$2\Delta = 0 \text{ or } \Delta = 0.$$

Example 14: *Without expanding the determinant, show that*

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \text{ and evaluate it.}$$

[UP TU B. Pharma 2001, 2008]

Solution: Let $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Multiplying the 1st, 2nd and 3rd rows by a, b, c respectively. We get

$$\Delta = \frac{1}{abc} \begin{vmatrix} a & a^2 & abc \\ b & b^2 & bca \\ c & c^2 & abc \end{vmatrix} = \frac{abc}{abc} \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Taking abc common from 3rd column

$$= \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} \text{ applying } C_2 \leftrightarrow C_3$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \text{ applying } C_1 \leftrightarrow C_2.$$

applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

on expanding the determinant along C_1

$$= (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

taking $(b-a)$ common from R_1 and $(c-a)$ common from R_2

$$= (b-a)(c-a)[c+a-(b+a)] = (b-a)(c-a)(c-b)$$
$$= (a-b)(b-c)(c-a).$$

Example 15: *Without expanding the determinant show that $(a + b + c)$ is a factor of following determinant.*

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$ **[UPTU B. Pharma 2003]**

If a, b, and c are positive and unequal, show that the value of Δ is always negative.

Solution: Applying $C_1 \to C_1 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$= (a+b+c)\begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix}$$

$$= (a+b+c)\{-(b-c)^2 - (a-b)(a-c)\}$$
$$= (a+b+c)(-a^2 - b^2 - c^2 + ab + bc + ca)$$

Thus $(a + b + c)$ is 0 factor of Δ. Now we shall prove the next part.

we have $\Delta = (a+b+c)(-a^2-b^2-c^2 + ab + bc + ca)$

$$= -\frac{1}{2}(a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$$

$$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

< 0, since a, b, c are positive and unequal. Therefore, the gives determinant is always negative if a, b, c are positive and unequal.

Example 16: *Show that*

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$$

Solution: Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + R_1$, we get

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} a & b & c \\ -b & -c & -a \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$

[Take $(a+b+c)$ out from R_3 and (-1) from R_2)

$$= -(a+b+c).\begin{vmatrix} a & b & c \\ b & c & a \\ 1 & 1 & 1 \end{vmatrix} \text{ (expand by } R_3)$$

$$= -(a+b+c).[1.(ab-c^2) - 1(a^2-bc) + 1.(ca-b^2)]$$
$$= -(a+b+c).(ab+bc+ca-a^2-b^2-c^2)$$
$$= -(a+b+c).(a^2+b^2+c^2-ab-bc-ca)$$
$$= a^3+b^3+c^3-3abc$$

Example 17: *Find the value of x if*

$$\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = 0$$

Solution: We have
$$\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = 0$$

Applying $(R_1 - R_3)$, we get

$$\begin{vmatrix} 1+x & 0 & -1-x \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = 0$$

Applying $C_3 \to C_3 + C_1$, we get

$$\begin{vmatrix} 1+x & 0 & 0 \\ 1 & 7+x & 7 \\ 2 & 5 & 5+x \end{vmatrix} = 0$$

On expanding the determinant along the first row, we get

$$(1+x)\begin{vmatrix} 7+x & 7 \\ 5 & 5+x \end{vmatrix} = 0$$

$$(1+x)[(7+x)(5+x)-35] = 0$$

or
$$(1+x)(x^2+12x) = 0$$
$$x(1+x)(x+12) = 0$$
$$x = 0, -1, -12.$$

Example 18: *Evaluate:*

$$|A| = \begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 23 & 39 & 8 & 38 \end{vmatrix}$$

Solution: Applying $C_1 \to C_1 - 3C_2$, $C_2 \to C_2 - 3C_3$, $C_4 \to C_4 - 4C_3$, we get

$$|A| = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 9 & 25 & 2 & 6 \\ 7 & 13 & 3 & 5 \\ 9 & 23 & 8 & 6 \end{vmatrix}$$

On expanding the determinant along first row, we get

$$= 1 \begin{vmatrix} 9 & 25 & 6 \\ 7 & 13 & 5 \\ 9 & 23 & 6 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$, we get

$$= 1 \begin{vmatrix} 0 & 2 & 0 \\ 7 & 13 & 5 \\ 9 & 23 & 6 \end{vmatrix}$$

On expanding the determinant along the first row, we get

$$= -2 \begin{vmatrix} 7 & 5 \\ 9 & 6 \end{vmatrix} = -2(42-45) = 6$$

Example 19: *Using properties of determinants, solve the following determinant for x.*

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix}$$

Solution: Given $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$ (operate $C_1 \to C_1 + C_2 + C_3$)

$$\Rightarrow \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} = 0$$

$$\Rightarrow (3a-x) \begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} = 0$$ (operate $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$)

$$\Rightarrow (3a-x) \begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 0$$ (expand by C_1)

$$\Rightarrow (3a-x).\,1.\begin{vmatrix} 2x & 0 \\ 0 & 2x \end{vmatrix} = 0$$

$$\Rightarrow (3a-x).\,(4x^2-0) = 0$$

$$\Rightarrow \quad 4x^2(3a-x) = 0$$

$$\Rightarrow \quad x^2 = 0 \text{ or } 3a - x = 0$$

$$\Rightarrow \quad x = 0, 0, 3a$$

Hence, the values of x are 0, 0, $3a$.

Example 20: *Using properties of determinants, prove that*

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$$

Solution: Operate $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \alpha & \beta-\alpha & \gamma-\alpha \\ \beta\gamma & \gamma(\alpha-\beta) & \beta(\alpha-\gamma) \end{vmatrix}$$

(Take $(\alpha-\beta)$ out from C_2 and $(\gamma-\alpha)$ out from C_3]

$$= (\alpha-\beta)(\gamma-\alpha) \begin{vmatrix} 1 & 0 & 0 \\ \alpha & -1 & 1 \\ \beta\gamma & \gamma & -\beta \end{vmatrix} \text{ (expand by } C_1)$$

$$= (\alpha-\beta)(\gamma-\alpha).1. \begin{vmatrix} -1 & 1 \\ \gamma & -\beta \end{vmatrix}$$

$$= (\alpha-\beta)(\gamma-\alpha)(\beta-\gamma)$$

$$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$$

Example 21: *Prove that*

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$$

Solution: We have $|A| = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$

Now multiply the column 1st, 2nd and 3rd by a, b and c respectively, we get

$$|A| = \frac{1}{abc} \begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & (c(c^2+1) \end{vmatrix}$$

To take a, b, c common from 1st, 2nd and 3rd rows respectively, we get

$$= \frac{abc}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

Now apply $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} a^2+b^2+c^2+1 & b^2 & c^2 \\ a^2+b^2+c^2+1 & b^2+1 & c^2 \\ a^2+b^2+c^2+1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (a^2+b^2+c^2+1) \begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Now applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= (a_2 + b_2 + c_2 + 1) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= a_2 + b_2 + c_2 + 1$$

$$= a_2 + b_2 + c_2 + 1$$

Example 22: *Prove that* $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

Solution: We have

$$\text{L.H.S.} = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 - 2C_3$, we get

$$= \begin{vmatrix} 2c & c+a & a+b \\ 2r & r+p & p+q \\ 2z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} c & c+a & a+b \\ r & r+p & p+q \\ z & z+x & x+y \end{vmatrix}$$

Now applying $C_2 \to C_2 - C_1$, we get

$$= 2 \begin{vmatrix} c & a & a+b \\ r & p & p+q \\ z & x & x+y \end{vmatrix}$$

Applying $C_3 \to C_3 - C_2$, we get

$= 2 \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ (by Interchanging the columns) = R.H.S.

Example 23: *If x,y, z are all different and* $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

show that $xyz = -1$.

Solution: Given $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

$$\Rightarrow \quad \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$

[Take x, y, z out from R_1, R_2 and R_3 respectively from the second determinant]

$$\Rightarrow \quad \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} (1+xyz) = 0$$

$$\Rightarrow \quad (x-y).(y-z).(z-x).(1+xyz) = 0$$

$$\Rightarrow \quad (1+xyz) = 0$$

(Because x, y, z are distinct, so $x-y \neq 0, y-z \neq 0, z-x \neq 0$).

$$\Rightarrow \quad xyz = -1$$

Example 24: *Evaluate the value of x for which*

$$\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$$

Solution: We have $\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$

Applying $\left(C_2 \to C_2 - \frac{3}{2}C_1\right)$ and $C_3 \to C_3 - 2C_1$, we get

$$\begin{vmatrix} 4x & 2 & 1 \\ 6x+2 & 0 & -4 \\ 8x+1 & -3/2 & 0 \end{vmatrix} = 0$$

Now applying $R_2 \to R_2 + 4R_1$

$$\Rightarrow \quad \begin{vmatrix} 4x & 2 & 1 \\ 22x+2 & 8 & 0 \\ 8x+1 & -3/2 & 0 \end{vmatrix} = 0$$

On expanding the determinants along 3rd column, we get

$$1\begin{vmatrix} 22x+2 & 8 \\ 8x+1 & -3/2 \end{vmatrix} = 0$$

$\Rightarrow$ $$-33x-3-64x-8 = 0$$

or $$-97x = 11 \text{ or } x = \frac{-11}{97}$$

Example 25: *Without expanding show that the value of the determinant given below is zero*

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$$

Solution: Let $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$

Using $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\cos\delta + \cos\alpha\sin\delta \\ \sin\beta & \cos\beta & \sin\beta\cos\delta + \cos\beta\sin\delta \\ \sin\gamma & \cos\gamma & \sin\gamma\cos\delta + \cos\gamma\sin\delta \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} \text{ Using } C_3 \to C_3 (\cos\delta) C_1 - (\sin\delta) C_2 = 0$$

Example 26: *Show that*

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a^2+b^2+c^2)(a+b+c)(b-c)(c-a)(a-l$$

Solution: Let $$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

Applying $C_1 \to C_1 - 2C_3$, we get

$$= \begin{vmatrix} b^2+c^2+a^2 & a^2 & bc \\ c^2+a^2+b^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix}$$

Operating $C_1 \to C_1 + C_2$, we get

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2-a^2 & (ca-bc) \\ 0 & c^2-a^2 & (ab-bc) \end{vmatrix}$$

$$= (a^2+b^2+c^2)(b-a)(c-a)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b+a & -c \\ 0 & c+a & -b \end{vmatrix}$$

$R_3 \to R_3 - R_2$, we get

$$= (a^2+b^2+c^2)(b-a)(c-a)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b+a & -c \\ 0 & c-a & c-b \end{vmatrix}$$

$$= (a^2+b^2+c^2)(b-a)(c-a)(c-b)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b+a & -c \\ 0 & 1 & 1 \end{vmatrix}$$

Expanding along first column, we get

$\Delta = (a^2+b^2+c^2)(b-a)(c-a)(c-b)(a+b+c)$

Example 27: *Show that*

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Solution: Let
$$\Delta = \begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & c+a & a+b \\ 2(a+b+a) & a+b & b+c \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & -a & -b \\ a+b+c & -b & -c \\ a+b+c & -c & -a \end{vmatrix}$$ Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

We get
$$= 2(-1)(-1)\begin{vmatrix} a+b+c & a & b \\ a+b+c & b & c \\ a+b+c & c & a \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2 - C_3$, we get

$$= 2\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$$

$$= 2\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} (C_1 \to C_2) = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & b & a \end{vmatrix}$$

Example 28: *If a, b, c (all positive) are the p^{th}, q^{th} and r^{th} terms respectively of a geometric progression, show that*

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$$

Solution: Consider the terms of G.P. which are $A, AR, AR^2, \ldots$.

$$a = T_p = AR^{p-1}$$
$$b = T_q = AR^{q-1}$$
$$c = T_r = AR^{r-1}$$

Consider $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = \begin{vmatrix} \log AR^{p-1} & p & 1 \\ \log AR^{q-1} & q & 1 \\ \log AR^{r-1} & r & 1 \end{vmatrix}$

$$= \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log A & p & 1 \\ \log A & q & 1 \\ \log A & r & 1 \end{vmatrix} + \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix}$$

$$= \log A \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} + \log R \begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix}$$

$$= \log A \times 0 + \log R \begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} = 0 + \log R \times 0 = 0$$

EXERCISE 3.1

Evaluate the following determinants (1 to 7):

1. $\begin{vmatrix} \frac{1}{2} & 8 \\ 4 & 2 \end{vmatrix}$

2. $\begin{vmatrix} -2 & 3 \\ 4 & -9 \end{vmatrix}$

3. $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

4. $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

5. $\begin{vmatrix} 1 & 0 & 6 \\ 3 & 4 & 15 \\ 5 & 6 & 21 \end{vmatrix}$

6. $\begin{vmatrix} 23 & 12 & 11 \\ 36 & 10 & 26 \\ 63 & 26 & 37 \end{vmatrix}$

7. $\begin{vmatrix} 3 & 1 & -4 \\ 3 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$

write the minor and co-factor of each element of the following determinants and also evaluate the determinants in each case (8 to 11):

8. $\begin{vmatrix} 5 & -10 \\ 0 & 3 \end{vmatrix}$

9. $\begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix}$

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

11. $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

12. Evaluate $\begin{vmatrix} x+1 & x+2 & x+4 \\ x+5 & x+6 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix}$

13. Evaluate $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

14. Evaluate $\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$

15. Evaluate $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

16. Prove that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(x-y)(z-x)$

17. Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

18. Prove that $\begin{vmatrix} x & y^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

19. Using properties of determinants, prove that

$$\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)(x-z)^2$$

20. Using properties of determinants, prove that

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

21. Solve the following determinants

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$ **[UP TU B. Pharma 2007]**

22. Prove that using properties of determinants

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

23. Prove that

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$

24. Prove that using properties of determinants

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$$

25. Prove that

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$$

HINTS TO THE SELECTED PROBLEMS

(i) $\begin{vmatrix} 1/2 & 8 \\ 4 & 2 \end{vmatrix} = \frac{1}{2} \times 2 - 8 \times 4 = 1 - 32 = -31.$

3. We have $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1.$

4. On expanding, we get $\Rightarrow (x^2 - x + 1)(x - 1) - (x - 1)(x + 1)$

$$= (x-1)(x^2 - x + 1 - x - 1)$$
$$= (x - 1)(x^2 - 2x)$$
$$= x^3 - 2x^2 - x^2 + 2x$$
$$|A| = x^3 - 3x^2 + 2x$$

5. $|A| = \begin{vmatrix} 1 & 0 & 6 \\ 3 & 4 & 15 \\ 5 & 6 & 21 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 3.2 \\ 3 & 4 & 3.5 \\ 5 & 6 & 3.7 \end{vmatrix} = 3\begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix}$

$$= 3\begin{vmatrix} 1 & 0 & 2 \\ 3 & 2.2 & 5 \\ 5 & 2.3 & 7 \end{vmatrix} = 6\begin{vmatrix} 1 & 0 & 2 \\ 3 & 2 & 5 \\ 5 & 3 & 7 \end{vmatrix} = 6\,[(14 - 15) + 2(9 - 10)] = -18.$$

8. $\begin{vmatrix} 5 & -10 \\ 0 & 3 \end{vmatrix}$

Minor of the element a_{11} is $M_{11} = |3| = 3$.

Minor of the element a_{12} is $M_{12} = 0$.

Minor of the element a_{21} is $M_{21} = -10$.

Minor of the element a_{22} is $M_{22} = 5$.

Hence cofactors are as $A_{11} = (-1)^{1+1} M_{11} = 3$

$$A_{12} = (-1)^{1+2} M_{12} = 0$$
$$A_{21} = (-1)^{2+1} M_{21} = 10$$
$$A_{22} = (-1)^{2+2} M_{22} = 5$$

$$|A| = \begin{vmatrix} 5 & -10 \\ 0 & 3 \end{vmatrix} = 15 - 0 = 15.$$

9. Minor of $a_{11} = \begin{vmatrix} -5 & 6 \\ 5 & 2 \end{vmatrix} = -40$ Minor of $a_{12} = \begin{vmatrix} 4 & 6 \\ 3 & 2 \end{vmatrix} = -10$

Minor of $a_{13} = \begin{vmatrix} 4 & -5 \\ 3 & 5 \end{vmatrix} = 35$ Minor of $a_{21} = \begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} = 16$

Minor of $a_{22} = \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 8$ Minor of $a_{23} = \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = -4$

Minor of $a_{31} = \begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = 8$ Minor of $a_{32} = \begin{vmatrix} 1 & -2 \\ 4 & 6 \end{vmatrix} = 14$

Minor of $a_{33} = \begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} = -17$

Now Cofactors are:

$A_{11} = (-1)^{1+1} M_{11} = -40$ $A_{31} = (-1)^{3+1} M_{31} = 8$

$A_{12} = (-1)^{1+2} M_{12} = 10$ $A_{32} = (-1)^{3+2} M_{32} = -14$

$A_{13} = (-1)^{1+3} M_{13} = 35$ $A_{33} = (-1)^{3+3} M_{33} = -17.$

$A_{21} = (-1)^{2+1} M_{21} = 16$
$A_{22} = (-1)^{2+2} M_{22} = 8$
$A_{23} = (-1)^{2+3} M_{23} = 4$

12. Applying the following operations and then expanding $R_3 \to R_3 - R_1, R_2 \to R_2 - R_1$ and $C_3 \to C_3 - C_1, C_2 \to C_2 - C_1$.

13. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$. And expanding along a_{11}, we get the required result.

14. Applying $R_1 \to R_1 + R_2 + R_3$.
Then $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ and expanding, we get the required result.

17. Taking *a*, *b*, *c* common from the first, second and third columns respectively, we get

$$\Delta = abc \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix} = a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Taking *a*,*b*, *c* common from 1st, 2nd and 3rd row respectively.

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} \text{ applying } R_2 \to R_2 + R_2, R_3 \to R_3 + R_1$$

$$= a^2b^2c^2(-1)(-4) = 4a^2b^2c^2.$$

18. Multiplying the first, second and third rows of the determinant on the L.H.S. by *x*, *y* and *z* respectively. We get,

$$= \frac{1}{xyz} \begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix}$$

$$= \frac{xyz}{xyz} \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, to the determinant. We get

$$= \begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2 - x^2 & z^2 - x^2 \\ x^3 & y^3 - x^3 & z^3 - x^3 \end{vmatrix}$$

$$= \begin{vmatrix} (y-x)(y+x) & (z-x)(z+x) \\ (y-x)(y^2 + xy + x^2) & (z-x)(z^2 + zx + x^2) \end{vmatrix}$$

$$= (y-x)(z-x) \begin{vmatrix} y+x & z+x \\ y^2 + xy + x^2 & z^2 + zx + x^2 \end{vmatrix}$$

[Taking $(y-x)$ common from the first column and $(z-x)$ from the second column]
Now Applying $C_2 \to C_2 - C_1$

We get
$$\Rightarrow (y-x)(z-x)\begin{vmatrix} y+x & z-y \\ y^2+xy+x^2 & (z^2-y2)+zx-xy \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} y+x & z-y \\ y^2+xy+x^2 & (z-y)(x+y+z) \end{vmatrix}$$

$$= (y-x)(z-x)(z-y)\begin{vmatrix} y+x & 1 \\ y^2+xy+x^2 & x+y+z \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx).$$

20. Applying $R_1 \to R_1 + R_2 + R_3$
$C_2 \to C_2 - C_1$
$C_3 \to C_2 - C_1$ we get the required result.

21. Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, the given equation becomes

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} = 0.$$

or
$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 0.$$

Expanding the determinant along the first row, the above equation becomes:
$(x-2)[30-24]-(2x-3)[10-6]+(3x-4)(4-3)=0$
$6x-12-8x+12+3x-4=0,\ x=4$

23.
$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}.$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & px^2 \\ 1 & y & py^2 \\ 1 & z & pz^2 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (1+pxyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (1+pxyz)(x-y)(y-z)(z-x).$$

25. $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta-\sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta-\sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta-\sin\gamma\sin\delta \end{vmatrix}$

Applying $C_3 \to C_3 + (\sin\delta)\,C_1 - (\cos\delta)\,C_2$. We get

$$\begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} = 0.$$

ANSWERS

1. -31 **2.** 6 **3.** 1 **4.** x^3-x^2+2

5. -18 **6.** 0 **7.** 49

8. $M_{11}=3, M_{12}=0, M_{21}=-10, M_{22}=5$
$A_{11}=3, A_{12}=0, A_{21}=10, A_{22}=5$, 15

9. $M_{11}=-40, M_{12}=-10, M_{13}=35, M_{21}=16, M_{22}=8,$
$M_{23}=-4, M_{31}=8.\ M_{32}=14, M_{33}=-17$
$A_{11}=-40, A_{12}=10, A_{13}=35, A_{21}=-16, A_{22}=8, A_{23}=4$
$A_{31}=8, A_{32}=-14, A_{33}=-17; -80$

10. $M_{11}=1, M_{12}=0, M_{13}=0, M_{21}=0, M_{22}=1, M_{23}=0,$
$M_{31}=0, M_{32}=0, M_{33}=1$
$A_{11}=1, A_{12}=0, A_{13}=0, A_{21}=0, A_{22}=1, A_{23}=0$
$A_{31}=0, A_{32}=0, A_{33}=1; 1$

11. $M_{11}=11, M_{12}=6, M_{13}=3, M_{21}=-4, M_{22}=2, M_{23}=1,$
$M_{31}=-20, M_{32}=-13, M_{33}=5$
$A_{11}=11, A_{12}=-6, A_{13}=3, A_{21}=4, A_{22}=2, A_{23}=-1,$
$A_{31}=20, A_{32}=13, A_{33}=5; 23$

12. -24 **13.** $(a-b)(b-c)(c-a)$

14. $\lambda^2(3x+\lambda)$ **15.** $4\,abc$ **21.** $x=4$.

3.6 CRAMER'S RULE

Consider the system of linear equations

$$a_1x+b_1y+c_1z = d_1$$
$$a_2x+b_2y+c_2z = d_2$$

$$a_2x + b_3y + c_3z = d_3 \qquad ...(1)$$

We define Δ = determinant coefficients

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Now we define Δ_x which is obtained by suppressing the column of coefficients of x and replacing it by the column of constant terms d_1, d_2, d_3 on right hand side

$$\therefore \qquad \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Similarly, we obtained

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

Now

Case I: If $\Delta \neq 0$ solution of system (1) is given by

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta}$$

and system is called consistent.

Case II: $\Delta = 0$ but at least one of Δ_x, Δ_y, $\Delta_z \neq 0$, then the system does not posses any common solution and system is called inconsistent.

Case III: $\Delta = 0$, also $\Delta_x = \Delta_y = \Delta_z = 0$ and at least one cofactor of $\Delta \neq 0$, then system has infinitely many solution and the system then be solved by elimination method.

Elimination of one unknown from three equations gives any one equations in two unknowns therefore two unknowns can be found in terms of the other, we give this unknown an arbitrary value.

If $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$ and all cofactor of Δ, Δx, Δy and Δz are zero then system is equivalent to only one equation in three unknowns and then we give any two unknowns arbitrary values and find the remaining unknown in terms of three constants.

SOLVED EXAMPLES

Example 1: *Using the Cramer's rule, solve the following system of equations*

$$x + y - 4 = 0,\ 2x - 3y - 8 = 0.$$

Solution: The given equation is

$$x + y - 4 = 0 \qquad ...(1)$$

$$2x - 3y - 8 = 0. \qquad ...(2)$$

Here, $$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5 \neq 0$$

$$\Delta_x = \begin{vmatrix} 4 & 1 \\ 3 & -3 \end{vmatrix} = -15$$

$$\Delta_y = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = -5.$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} = 3, y = \frac{\Delta_y}{\Delta} = 1.$$

Example 2: *Show that the system of equations $x + y - 2 = 0$. $2x + 3y - 5 = 0$, $4x - y - 3 = 0$ is consistent. Find the solution using Cramer's rule.*

Solution: The given system of equation is

$$x + y - 2 = 0$$
$$2x + 3y - 5 = 0$$
$$4x - y - 3 = 0$$

is consistent (i.e., have common solution), if the determinant

$$\Delta^* = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 3 & -5 \\ 4 & -1 & -3 \end{vmatrix} = 0 \text{ i.e., } \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 4 & -5 & 5 \end{vmatrix} = 0.$$

$\therefore$ $\Delta^* = 5 - 5 = 0$, hence the system is consistent, so it is sufficient to solve any two equations by Cramer's rule.

Let us consider equation (1) and (2)

$$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \ (\neq 0)$$

$$\Delta_x = \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 6 - 5 = 1$$

$$\Delta_y = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1$$

$$x = \frac{\Delta_x}{\Delta} = 1 \ y = \frac{\Delta_y}{\Delta} = 1$$

Hence the required solution is given by $x = y = 1$.

Example 3: *Solve the following by Cramer's rule*

$$x + y + z = 6$$
$$x - y + z = 2$$
$$3x + 2y - 4z = -5.$$

Solution: We have $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 3 & -1 & 7 \end{vmatrix} = 14 \neq 0$

$$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ -5 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 1 \\ -4 & -2 & 0 \\ 19 & 6 & 0 \end{vmatrix} = 14$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 3 & -5 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 1 \\ 0 & -4 & 0 \\ 0 & -23 & -7 \end{vmatrix} = 28$$

$$\Delta = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 3 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 6 \\ 0 & -2 & -4 \\ 0 & -1 & -23 \end{vmatrix} = 42$$

Hence, by Cramer's rule

$$x = \frac{\Delta_x}{\Delta} = 1, \frac{\Delta_y}{\Delta} = 2 = y, z = \frac{\Delta_z}{\Delta} = 3.$$

Hence, the solution is given by $x = 1, y = 2, z = 3$.

Example 4: *Solve the following system equations with the help of Cramer's rule.*

$3x - 4y + 5z = -6$, $x + y - 2z = -1$, $2x + 3y + z = 5$. **[UPTU B. Pharma 2004]**

Solution: Let $\Delta = \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 3(1+6) + 4(1+4) + 5(3-2) = 46 \neq 0.$

since $\Delta \neq 0$, therefore the given system has a unique solution given by

$$\frac{x}{\Delta_x} = \frac{y}{\Delta_x} = \frac{z}{\Delta_z} = \frac{1}{\Delta}.$$

Now $\Delta_x = \begin{vmatrix} -6 & -4 & 5 \\ -1 & 1 & -2 \\ 5 & 3 & 1 \end{vmatrix}$ by $R_1 \to R_1 + 4R_2, R_3 \to R_3 - 3R_2$.

$$= \begin{vmatrix} -10 & 0 & -3 \\ -1 & 1 & -2 \\ 8 & 0 & 7 \end{vmatrix} = \begin{vmatrix} -10 & -3 \\ 8 & 7 \end{vmatrix} = -70 + 24 = -46.$$

$\Delta_y = \begin{vmatrix} 3 & -6 & 5 \\ 1 & -1 & -2 \\ 2 & 5 & 1 \end{vmatrix}$ by $R_1 \to R_1 - 3R_2, R_3 \to R_3 - 2R_2$

$$= \begin{vmatrix} 0 & -3 & 11 \\ 1 & -1 & -2 \\ 0 & 7 & 5 \end{vmatrix} = -\begin{vmatrix} -3 & 11 \\ 7 & 5 \end{vmatrix} = 92$$

$$\Delta_z = \begin{vmatrix} 3 & -4 & -6 \\ 1 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix} \text{ by } R_1 \to R_1 - 3R_2, R_3 \to R_3 - 2R_2$$

$$= \begin{vmatrix} 0 & -7 & -3 \\ 1 & 1 & -1 \\ 0 & 1 & 7 \end{vmatrix} = -\begin{vmatrix} -7 & -3 \\ 1 & 7 \end{vmatrix} = 46$$

The solution of the given system is

$$x = \frac{\Delta_x}{\Delta} = \frac{-46}{46} = -1, y = \frac{\Delta_y}{\Delta} = \frac{92}{46} = 2 \text{ and } z = \frac{\Delta_z}{\Delta} = \frac{46}{46} = 1$$

Hence, the required solution is $x = -1, y = 2, z = 1$.

Example 5: *Solve using Cramer's rule*

$x + y = 5, y + z = 3, z + x = 4.$ **[UPTU B. Pharma 2001, 07]**

Solution: Let $\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1(1-0) - 1(0-1) = 1 + 1 = 2$

since $\Delta \neq 0$, therefore the given systems has a unique solution given by

$$\frac{x}{\Delta_x} = \frac{y}{\Delta_y} = \frac{z}{\Delta_z} = \frac{1}{\Delta}$$

Now $\Delta_x = \begin{vmatrix} 5 & 1 & 0 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{vmatrix} = 6.$

$$\Delta_y = \begin{vmatrix} 1 & 5 & 0 \\ 0 & 3 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 4$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix} = 2.$$

The solution of the given system is

$$x = \frac{\Delta_x}{\Delta} = \frac{6}{2} = 3, y = \frac{\Delta_y}{\Delta} = \frac{4}{2} = 2, z = \frac{\Delta_z}{\Delta} = \frac{2}{2} = 1.$$

Example 6: *Solve the following by using Cramer's rule*

$x - 2y + 3z = 2, 2x - 3z = 3, x + y + z = 6.$ **[UPTU B. Pharma 2002]**

Solution: Let $\Delta = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 4 & -9 \\ 1 & 3 & -2 \end{vmatrix}$ by $R_2 + 2R_1, R_3 - 3R_1$

$$= \begin{vmatrix} 4 & -9 \\ 3 & -2 \end{vmatrix} = -8 + 27 = 19 \neq 0$$

since $\Delta \neq 0$, therefore the given system has a unique solution given by

$$\frac{x}{\Delta_x} = \frac{y}{\Delta_y} = \frac{z}{\Delta_z} = \frac{1}{\Delta}.$$

Now
$$\Delta_x = \begin{vmatrix} 2 & -2 & 3 \\ 3 & 0 & -3 \\ 6 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 5 \\ 3 & 3 & 0 \\ 6 & 7 & 7 \end{vmatrix} \text{ by } R_2 + R_1, R_3 + R_1$$

$$= 2\begin{vmatrix} 3 & 0 \\ 7 & 7 \end{vmatrix} + 0 + 5\begin{vmatrix} 3 & 3 \\ 6 & 7 \end{vmatrix} = 57.$$

$$\Delta_y = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -3 \\ 1 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -9 \\ 1 & 4 & -2 \end{vmatrix} \text{ by } R_2 - 2R_1, R_3 - 3R_1$$

$$= 1\begin{vmatrix} -1 & -9 \\ 4 & -2 \end{vmatrix} = 38.$$

$$\Delta_z = \begin{vmatrix} 1 & -2 & 2 \\ 2 & 0 & 3 \\ 1 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 4 & -1 \\ 1 & 3 & 4 \end{vmatrix} \text{ by } R_2 + 2R_1, R_3 - 2R_1$$

$$= 1\begin{vmatrix} 4 & -1 \\ 3 & 4 \end{vmatrix} = 16 + 3 = 19.$$

The solution of the given system is

$$x = \frac{\Delta_x}{\Delta} = \frac{57}{19} = 3, y = \frac{\Delta_y}{\Delta} = \frac{38}{19} = 1 \text{ and } z = \frac{\Delta_z}{\Delta} = \frac{19}{19} = 1.$$

Hence, the required solution is $x = 3, y = 2, z = 1$.

Example 7: *Find the value of λ for which the system of equations $x + y - 2z = 0$, $2x - 3xy + z = 0$, $x - 5y + 4z = \lambda$ are consistent and find the solutions for all such values of λ.*

Solution: The given system of equations is

$$x - 5y + 4z = \lambda \quad \text{...(1)}$$
$$x + y - 2z = 0 \quad \text{...(2)}$$
$$2x - 3y + z = 0. \quad \text{...(3)}$$

$$\Delta = \begin{vmatrix} 1 & -5 & 4 \\ 1 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -5 & 4 \\ 0 & 6 & -6 \\ 0 & 7 & -7 \end{vmatrix} = 0.$$

Hence, system is consistent only when

$$\Delta_x = \Delta_y = \Delta_z = 0.$$

Now
$$\Delta_x = \begin{vmatrix} \lambda & -5 & 4 \\ 0 & 1 & -2 \\ 0 & -3 & 1 \end{vmatrix} = -5\lambda = 0 \Rightarrow \lambda = 0.$$

For $\lambda = 0$, clearly $\Delta_y = \Delta_z = 0$.

$\therefore$ System is consistent if $\lambda = 0$, then on eliminating x from (1), (2) and (1), (3), we have

$6y - 6z = 0, y - z = 0$

and $7y - 7z = 0$ or $y = z$.

Let $y = z = k \in R$, then from (1), we have $x = 5k - 4k = k$.

Hence, solution is given by $x = y = z = k \in R$.

Example 8: *Solve the equations by Cramer's rule*

$$\frac{4}{x+5} + \frac{3}{y+7} = -1.$$

$$\frac{6}{x+5} + \frac{6}{y+7} = -5.$$

Solution: The given system of equations is

$$\frac{4}{x+5} + \frac{3}{y+7} = -1$$

$$\frac{6}{x+5} + \frac{6}{y+7} = -5.$$

Now putting $\dfrac{1}{x+5} = a, \dfrac{1}{y+7} = b$, the equation becomes

$$4a + 3b = -1$$

$$6a - 6b = -5$$

$$\Delta = \begin{vmatrix} 4 & 3 \\ 6 & -6 \end{vmatrix} = -42 \neq 0$$

$$\Delta_a = \begin{vmatrix} -1 & 3 \\ 5 & -6 \end{vmatrix} = 21,\ \Delta_b = \begin{vmatrix} 4 & -1 \\ 6 & -5 \end{vmatrix} = -14.$$

So by Cramer's rule

$$a = \frac{\Delta_a}{\Delta} = \frac{21}{-42} = \frac{1}{2}, b = \frac{\Delta_b}{\Delta} = \frac{-14}{-42} = \frac{1}{3}$$

$\therefore \quad x + 5 = -2, y + 7 = 3$

or $\quad x = -7, y = -4$.

Hence, the solution is $x = -7, y = -4$.

Example 9: *Using Cramer's rule solve the following equations:*

$$x + 2y + 3z = 6$$

$$2x + 4y + z = 17$$

$$3x + 2y + 9z = 2$$

Solution: We have

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 4 & 9 \end{vmatrix} = -20$$

$$\Delta_x = \begin{vmatrix} 6 & 2 & 3 \\ 17 & 4 & 1 \\ 2 & 2 & 9 \end{vmatrix} = -20$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 17 & 1 \\ 3 & 2 & 9 \end{vmatrix} = -80$$

$$\Delta_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 17 \\ 3 & 2 & 2 \end{vmatrix} = -20$$

Then by Cramer's rule, we have

$$x = \frac{\Delta_x}{\Delta} = \frac{-20}{-20} = 1$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-80}{-20} = 4$$

$$z = \frac{\Delta_z}{\Delta} = \frac{20}{-20} = -1$$

EXERCISE 3.2

1. (a) Using Cramer's rule, solve the following equations

$$x + y + z = 6$$
$$2x + y - z = 1$$
$$x + y - 2z = -3.$$

(b)

$$x + y + z = 6$$
$$x - y + z = 2$$
$$3x + 2y - 9z = -5$$

2. Find the value of k if the following equation are consistent:

$$x + y - 3 = 0$$
$$(1 + k)x + (2 + k)y - 8 = 0$$
$$x - (1 + k)y + (2 + k) = 0.$$

3. Find the value of k if the system of equations

$$(k+1)^3 x + (k+2)^3 y = (k+3)^3$$
$$(k+1)x + (k+2)y = (k+3)$$
$$x + y = 1 \text{ : is consistent.}$$

4. If the system of equations

$x + 2y = 5, 2x - y = 5, x + 3y = 6$ is consistent, solve it.

5. Solve the following by Cramer's rule

$$x + y + z = 11$$
$$2x - 6y - z = 0$$
$$3x + 4y + 2z = 0.$$

6. Show that the system of equation

$$3x - y + 4z = 3$$
$$x + 2y - 3z = -2$$
$$6x + 5y + \lambda z = -3$$

has at least one solution for any real numbers λ. Find the set of solution if $\lambda = -5$.

7. Using Cramer's rule to solve the following system of linear equations

$$2x - 3y + z = 7$$
$$2x + y + z = 1$$
$$4y + 3z = -11$$

ANSWERS

1. (a) $x = 1, y = 2, z = 3$ (b) $x = \frac{9}{4}, y = 2, z = \frac{21}{12}$

2. $k = 1$ or $-5/3$ **3.** $k = -2$ **5.** $x = -8, y = -7, z = 26$.

REFRESHER

Do you know? *After reading this chapter you should be able to learn the following concepts:*

- The value of a determinant does not change when rows and columns are interchanged.
- If any two rows (or columns) of a determinant are inter changed, the sign of the determinant in changed.
- If two rows or two columns of the determinant are identical, then the value of the determinant are vanishes.
- If all the elements of any row, or any column, of a determinant are multiplied by the same number, then the determinant is multiplied by that number.
- If in a determinant each element in any row (or column) consists of two terms, then the determinant can be expressed as the sum of two other determinant.
- If in a determinant, the elements of a row are added in and n times the corresponding elements of the another rows (or columns) the value of determinant does not change is particular

$$\begin{vmatrix} a_1 + mb_1 + nc_1 & b_1 & c_1 \\ a_2 + mb_2 + nc_2 & b_2 & c_2 \\ a_3 + mb_3 + nc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix}$$

Can we do? (Frequently Asked Questions)

1. Without expanding the determinant, show that

$$\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0.$$ **[UPTU B. Pharma 2002]**

2. Without expanding the determinant, show that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ and evaluate it. **[UPTU B. Pharma 2001]**

3. Prove that

$$\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$$= abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = abc + ab + bc + ca.$$ **[UPTU B. Pharma 2004, 06]**

4. Without expanding the determinant show that $(a + b + c)$ is 0 factor of the following determinant

$$\Delta \equiv \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

If a, b, c are positive and unequal. Show that the value of Δ is always negative. **[UPTU B. Pharma 2003]**

5. Solve the equations $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$ **[UPTU B. Pharma 2007]**

6. Prove that $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$ **[UPTU B. Pharma 2005]**

7. Solve by Cramer's rule
$x - 2y = 4, -3x + y = -7.$ **[UPTU B. Pharma 2001]**

8. Solve using Cramer's rule
$x + y = 5, y + z = 3, z + x = 4.$ **[UPTU B. Pharma 2001, 07]**

9. Solve the following by using Cramer's rule.
$x - 2y + 3z = 2, 2x - 3z = 3, x + y + z = 6.$ **[UPTU B. Pharma 2002]**

10. Solve the following system of linear equations with the help of Cramer's rule.
$3x - 4y + 5y = 6, x + y - 2z = -1, 2x + 3y + z = 5.$ **[UPTU B. Pharma 2004]**

11. Prove that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$ **[UPTU B. Pharma 2008]**

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